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DISCRETE COUNTERPARTS OF CONTINUOUS-TIME ADDITIVE HOPFIELD-TYPE NEURAL NETWORKS WITH IMPULSES

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ABSTRACT. Discrete-time analogues of continuous-time additive Hopfield-type neural networks with impulses are formulated and their global stability characteristics are investigated.
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1. INTRODUCTION

In the modelling and analysis of dynamical phenomena various types of systems ranging downward in complexity from partial differential equations, functional differential equations, integro-differential equations, stochastic differential equations with hereditary term, difference equations and algebraic equations have been used. It is common to approximate models of higher levels of complexity by models of lower levels of complexity.

One of the most widely used techniques in the study of models involving ordinary differential equations is to approximate the system by means of a system of difference equations, whose solutions are expected to be samples of the solutions of differential equations at discrete instants of time as in the case of Euler-type methods and Runge-Kutta methods. It has been shown by several authors (Mickens, 1994; Prufer, 1985) that the dynamics of numerical discretizations of differential equations can differ significantly from those of the original differential equations. Examples comparing the dynamics of the respective pairs of differential and difference equations are given in (Mohamad & Gopalsamy, 2000)(see also the introduction of our previous work (Akça et al., submitted)).

In (Mohamad & Gopalsamy, 2000) the global stability characteristic of a system of equations modelling the dynamics of additive Hopfield-type neural networks both in the continuous and discrete time cases is investigated. In particular, a novel method of obtaining a discrete-time dynamical system whose dynamics is inherited from the continuous-time dynamical system is studied. This aspect is important since numerical algorithms of Hopfield-type differential equations lead to discrete-time dynamic systems and such discrete-time systems should not give rise to any spurious behaviour if

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either system is to be used for coding equilibrium as associative memories corresponding to temporally uniform external stimuli obtained. The discrete-time models serve as global numerical methods on unbounded intervals for the continuous-time systems (Meinardus & Nurnberger, 1985).

In our previous paper we investigated the global stability characteristics of these systems supplemented with impulse conditions in the continuous-time case. The presence of impulses required some modifications and the imposing of additional conditions on the systems. The present paper is devoted to the formulation of the discrete-time analogues of these impulsive systems and the investigation of their stability. Let us recall that convergent difference approximations for nonlinear impulsive systems of differential equations in a Banach space were obtained in (Covachev et al., 2001).

2. MAIN RESULTS

Consider the following Hopfield-type model of neural network with impulses

$$\frac{dx_i}{dt} = -a_i x_i(t) + \sum_{j=1}^{m} b_{ij} f_j(x_j(t)) + c_i, \quad t > 0, \quad t \neq t_k, \tag{2.1}$$

$$\Delta x_i(t_k) = I_i(x_i(t_k)), \qquad i = \overline{1,m}, \qquad k = 1, 2, \ldots, \tag{2.2}$$

where $\Delta x(t_k) = x(t_k + 0) - x(t_k - 0)$ are the impulses at moments t_k and $t_1 < t_2 < \ldots$ is a strictly increasing sequence such that $\lim_{k\to\infty} t_k = +\infty$; $x_i(t)$ corresponds to the membrane potential of the unit i at time t; $f_j(\cdot)$ denotes a measure of response or activation to its incoming potentials; b_{ij} denotes the synaptic connection weight of the unit j on the unit i; the constants c_i correspond to the external bias or input from outside the network to the unit i; the coefficient a_i is the rate with which the unit self-regulates or resets its potential when isolated from other units and inputs. We refer for more details about neural networks to (Mohamad & Gopalsamy, 2000; Mohamad, 2000; Akça et al., submitted) and the references cited therein.

As usual in the theory of impulsive differential equations, at the points of discontinuity t_k of the solution $t \mapsto x_i(t)$ we assume that $x_i(t_k) \equiv x_i(t_k - 0)$. It is clear that, in general, the derivatives $\dot{x}_i(t_k)$ do not exist. On the other hand, according to the equality (2.1) there exist the limits $\dot{x}_i(t_k \mp 0)$. According to the above convention, we assume $\dot{x}_i(t_k) \equiv \dot{x}_i(t_k - 0)$.

The system (2.1) was supposed to satisfy the following conditions

H1.: $f_i : \mathbb{R} \to \mathbb{R}$ is Lipschitzian with Lipschitz constant $L_i > 0$,

$$|f_i(x) - f_i(y)| \leq L_i |x - y| \quad \text{for all} \quad x, y \in \mathbb{R}.$$

H2.: $|f_i(x)| \leq M_i, \quad x \in \mathbb{R}$, for some constant $M_i > 0$.

H3.: $a_i > 0, \quad b_{ij} \in \mathbb{R}, \quad c_i \in \mathbb{R}, \quad i, j \in \{1, \ldots, m\}$.

H4.: The following inequalities hold:

$$a_i - L_i \sum_{j=1}^{m} |b_{ji}| > 0, \qquad i = \overline{1,m}.$$

System (2.1) has a unique equilibrium $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$ whose components obviously satisfy

$$a_i x_i^* = \sum_{j=1}^{m} b_{ij} f_j(x_j^*) + c_i, \qquad i = \overline{1,m}. \tag{2.3}$$

either system is to be used for coding equilibrium as associative memories corresponding to temporally uniform external stimuli obtained. The discrete-time models serve as global numerical methods on unbounded intervals for the continuous-time systems (Meinardus & Nurnberger, 1985).

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System (2.1) has a unique equilibrium $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$ whose components obviously satisfy

$$a_i x_i^* = \sum_{j=1}^{m} b_{ij} f_j(x_j^*) + c_i, \qquad i = \overline{1,m}. \tag{2.3}$$

Moreover, the impulsive operators $I_i(x_i(t))$ in (2.2) were supposed to satisfy

$$I_i(x_i(t_k)) = -\gamma_{ik}(x_i(t_k) - x_i^*), \quad 0 < \gamma_{ik} < 2, \quad i = \overline{1,m}, \quad k \in \mathbb{Z}^+. \tag{2.4}$$

Here and below we use the notation $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, $\mathbb{Z}_0^+ = \{0, 1, 2, \ldots\}$, $\mathbb{Z}_0^- = \{\ldots, -2, -1, 0\}$.

In (Akça et al., submitted) all solutions of (2.1), (2.2) were proved to satisfy the inequality

$$\sum_{i=1}^{m} |x_i(t) - x_i^*| \le e^{-\alpha t} \sum_{i=1}^{m} |x_i(0) - x_i^*|, \qquad t > 0,$$

for some constant $\alpha > 0$.

Following (Mohamad & Gopalsamy, 2000), we reformulate system (2.1) by an approximation of the form

$$\frac{dx_i(s)}{ds} = -a_i x_i(s) + \sum_{j=1}^{m} b_{ij} f_j(x_j([t/h]h)) + c_i, \quad i = \overline{1,m}, \; s \in [[t/h]h, [t/h]h + h), \tag{2.5}$$

where h is a positive number denoting a uniform discretization step size and $[t/h]$ denotes the greatest integer in t/h. For convenience, we denote $[t/h] = n$, $n \in \mathbb{Z}_0^+$, and $x_i(nh) = x_i(n)$. Thus (2.5) takes on the form

$$\frac{dx_i(s)}{ds} = -a_i x_i(s) + \sum_{j=1}^{m} b_{ij} f_j(x_j(n)) + c_i, \quad i = \overline{1,m}, \quad s \in [nh, (n+1)h). \tag{2.6}$$

In (Mohamad & Gopalsamy, 2000) the step h is an arbitrary positive number and the approximation (2.6) of equation (2.1) is used for all $n \in \mathbb{Z}_0^+$. However, we shall use this approximation only for nonnegative integers n such that the interval $[nh, (n+1)h)$ contains no moment of impulse effect t_k, $k \in \mathbb{Z}^+$.

We can rewrite the equation (2.6) in the form

$$\frac{d}{ds}\left(x_i(s)e^{a_i s}\right) = e^{a_i s}\left(\sum_{j=1}^{m} b_{ij} f_j(x_j(n)) + c_i\right), \quad i = \overline{1,m}, \; s \in [nh, (n+1)h),$$

and integrate it over the interval $[nh, t]$ for $t < (n+1)h$ to obtain

$$x_i(t)e^{a_i t} - x_i(n)e^{a_i nh} = \frac{e^{a_i t} - e^{a_i nh}}{a_i}\left(\sum_{j=1}^{m} b_{ij} f_j(x_j(n)) + c_i\right), \quad i = \overline{1,m}.$$

In the last equality we let $t \to (n+1)h$ and obtain

$$x_i(n+1) = e^{-a_i h} x_i(n) + \frac{1 - e^{-a_i h}}{a_i}\left(\sum_{j=1}^{m} b_{ij} f_j(x_j(n)) + c_i\right), \quad i = \overline{1,m}. \tag{2.7}$$

This system is the discrete-time analogue of the system (2.1). An equilibrium $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$ of (2.7) satisfies the system

$$\frac{1 - e^{-a_i h}}{a_i}\left\{a_i x_i^* - \left(\sum_{j=1}^{m} b_{ij} f_j(x_j^*) + c_i\right)\right\} = 0, \quad i = \overline{1,m}. \tag{2.8}$$

Obviously the quantities

$$\phi_i(h) = \frac{1 - e^{-a_i h}}{a_i}, \qquad i = \overline{1,m},$$

satisfy $\phi_i(h) > 0$, thus (2.8) implies (2.3), *i.e.*, the equilibria of the systems (2.1) and (2.7) coincide. We write the system (2.7) in the form

$$x_i(n+1) = e^{-a_i h} x_i(n) + \phi_i(h) \sum_{j=1}^{m} b_{ij} f_j(x_j(n)) + \phi_i(h) c_i, \quad i = \overline{1,m}. \tag{2.9}$$

In (Mohamad & Gopalsamy, 2000) no restrictions were imposed on the step size h. Neither were such restrictions required to obtain the stability result for system (2.9).

However, we are investigating the impulsive system (2.1), (2.2). We find it appropriate to assume there is not more than one moment of impulse effect in a step. To this end we suppose that

$$\theta = \inf_{k \in \mathbb{Z}_0^+} (t_{k+1} - t_k) > 0 \tag{2.10}$$

and $h > 0$ satisfies

$$h < \theta. \tag{2.11}$$

We denote $[t_k/h] = n_k$, $k \in \mathbb{Z}^+$, and replace system (2.1), (2.2) satisfying (2.4) by the following discrete system

$$\begin{aligned} x_i(n+1) &= e^{-a_i h} x_i(n) + \phi_i(h) \sum_{j=1}^{m} b_{ij} f_j(x_j(n)) + \phi_i(h) c_i, \\ &\qquad n \in \mathbb{Z}_0^+ \setminus \{n_1, n_2, \ldots\}, && (2.12) \\ x_i(n_k+1) &= x_i(n_k) - \gamma_{ik}(x_i(n_k) - x_i^*), && (2.13) \\ &\qquad 0 < \gamma_{ik} < 2, \quad k \in \mathbb{Z}^+, \quad i = \overline{1,m}. \end{aligned}$$

Theorem 1.

Let $h > 0$ satisfy (2.11). Suppose the assumptions **H1**–**H4** hold. Then there exists a positive number $\rho = \rho(h) < 1$ such that for any compact neighbourhood K of the equilibrium $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$ there exists a constant $C \geq 1$ such that all solutions $x(n)$ of (2.12), (2.13) with $x(0) \in K$ satisfy the following inequality

$$\sum_{i=1}^{m} \frac{|x_i(n+1) - x_i^*|}{\phi_i(h)} \leq C\rho^n \sum_{i=1}^{m} \frac{|x_i(0) - x_i^*|}{\phi_i(h)}, \quad n \in \mathbb{Z}^+.$$

Proof. The proof of the theorem for $n \neq n_k$, $k \in \mathbb{Z}^+$, follows like that of (Mohamad & Gopalsamy, 2000, Theorem 4.1). We have from (2.12) that

$$x_i(n+1) - x_i^* = e^{-a_i h}(x_i(n) - x_i^*) + \phi_i(h) \sum_{j=1}^{m} b_{ij}(f_j(x_j(n)) - f_j(x_j^*))$$

for $i = \overline{1,m}$, $n \in \mathbb{Z}_0^+ \setminus \{n_1, n_2, \ldots\}$, and hence by condition **H1**

$$|x_i(n+1) - x_i^*| \leq e^{-a_i h}|x_i(n) - x_i^*| + \phi_i(h) \sum_{j=1}^{m} |b_{ij}|\, L_j\, |x_j(n) - x_j^*|$$

for $i = \overline{1,m}$, $n \in \mathbb{Z}_0^+$, $n \neq n_k$, $k \in \mathbb{Z}^+$. We rewrite this inequality as

$$\frac{|x_i(n+1) - x_i^*|}{\phi_i(h)} \leq e^{-a_i h} \frac{|x_i(n) - x_i^*|}{\phi_i(h)} + \sum_{j=1}^{m} |b_{ij}|\, L_j\, \phi_j(h) \frac{|x_j(n) - x_j^*|}{\phi_j(h)}, \quad n \neq n_k,$$

and obtain

$$\sum_{i=1}^{m}\frac{|x_i(n+1)-x_i^*|}{\phi_i(h)} \le \sum_{i=1}^{m}\left\{e^{-a_ih}\frac{|x_i(n)-x_i^*|}{\phi_i(h)}+\sum_{j=1}^{m}|b_{ij}|\,L_j\,\phi_j(h)\frac{|x_j(n)-x_j^*|}{\phi_j(h)}\right\}$$

$$= \sum_{i=1}^{m}\left\{e^{-a_ih}+L_i\phi_i(h)\sum_{j=1}^{m}|b_{ji}|\right\}\frac{|x_i(n)-x_i^*|}{\phi_i(h)} \tag{2.14}$$

for $n \in \mathbb{Z}_0^+ \setminus \{n_1, n_2, \dots\}$. Now, for each $i = \overline{1,m}$ we have

$$0 < e^{-a_ih}+L_i\phi_i(h)\sum_{j=1}^{m}|b_{ji}| = 1-\phi(h)\left(a_i-L_i\sum_{j=1}^{m}|b_{ji}|\right).$$

By virtue of condition **H4** there exists a real number

$$\alpha = \min_{i=\overline{1,m}}\left(a_i-L_i\sum_{j=1}^{m}|b_{ji}|\right) > 0.$$

We also denote

$$\alpha' = \alpha'(h) = \min_{i=\overline{1,m}}\phi_i(h) > 0$$

and $\rho = \rho(h) = 1-\alpha\alpha'(h)$. Now

$$0 < e^{-a_ih}+L_i\phi_i(h)\sum_{j=1}^{m}|b_{ji}| \le \rho < 1$$

and inequality (2.14) takes on the form

$$\sum_{i=1}^{m}\frac{|x_i(n+1)-x_i^*|}{\phi_i(h)} \le \rho\sum_{i=1}^{m}\frac{|x_i(n)-x_i^*|}{\phi_i(h)}, \quad n\in\mathbb{Z}_0^+\setminus\{n_1,n_2,\dots\}. \tag{2.15}$$

On the other hand, from (2.13) we have

$$|x_i(n_k+1)-x_i^*| = |1-\gamma_{ik}|\,|x_i(n_k)-x_i^*|, \quad i=\overline{1,m}, \quad k\in\mathbb{Z}^+. \tag{2.16}$$

Let us denote $\gamma_k = \max_{i=\overline{1,m}}|1-\gamma_{ik}|$ (obviously $0<\gamma_k<1$) and $\gamma = \sup_{k\in\mathbb{Z}^+}\gamma_k$ (thus $0<\gamma\le 1$). Now the equalities (2.16) imply

$$|x_i(n_k+1)-x_i^*| \le \gamma\,|x_i(n_k)-x_i^*|, \quad i=\overline{1,m}, \quad k\in\mathbb{Z}^+,$$

and

$$\sum_{i=1}^{m}\frac{|x_i(n_k+1)-x_i^*|}{\phi_i(h)} \le \gamma\sum_{i=1}^{m}\frac{|x_i(n_k)-x_i^*|}{\phi_i(h)}, \quad k\in\mathbb{Z}^+. \tag{2.17}$$

By virtue of (2.15) and (2.17) we have

$$\sum_{i=1}^{m}\frac{|x_i(n)-x_i^*|}{\phi_i(h)} \le \rho^{n-i(1,n)}\gamma^{i(1,n)}\sum_{i=1}^{m}\frac{|x_i(0)-x_i^*|}{\phi_i(h)}, \quad n\in\mathbb{Z}^+, \tag{2.18}$$

where $i(1,n)$ is the number of n_k among $1,2,\dots,n$. Now we consider two cases:

a) $\gamma<1$. Put $\bar\rho = \max(\rho,\gamma)<1$. Thus (2.18) takes on the form

$$\sum_{i=1}^{m}\frac{|x_i(n)-x_i^*|}{\phi_i(h)} \le \bar\rho^{n}\sum_{i=1}^{m}\frac{|x_i(0)-x_i^*|}{\phi_i(h)}, \quad n\in\mathbb{Z}^+.$$

b) $\gamma = 1$. Here we introduce some notation to be repeatedly used henceforth. Let us denote by $i(0,t)$ the number of moments of impulse effect t_k in the interval $(0,t)$. Further on, denote

$$p = \limsup_{t\to\infty} \frac{i(0,t)}{t}, \qquad q = \limsup_{n\to\infty} \frac{i(1,n)}{n}.$$

Obviously, we have $q \leq 1$. By virtue of (2.10) $p \leq 1/\theta$ (and thus $p < \infty$). Moreover, it is easily seen that $q = ph$ and since $h < \theta$, then $q < 1$.

There exists $\eta > 0$ such that $q + \eta < 1$. We have $i(1,n) \leq (q+\eta)n$ for n large enough. Now

$$\rho^{n-i(1,n)} \leq \rho^{n-(q+\eta)n} = \{\rho^{1-(q+\eta)}\}^n$$

for n large enough. If we denote $\tilde{\rho} = \rho^{1-(q+\eta)} < 1$, then from (2.18) we obtain

$$\sum_{i=1}^{m} \frac{|x_i(n)-x_i^*|}{\phi_i(h)} \leq \tilde{\rho}^n \sum_{i=1}^{m} \frac{|x_i(0)-x_i^*|}{\phi_i(h)}$$

for n large enough.

There exists a constant $C \geq 1$ such that

$$\sum_{i=1}^{m} \frac{|x_i(n)-x_i^*|}{\phi_i(h)} \leq C\tilde{\rho}^n \sum_{i=1}^{m} \frac{|x_i(0)-x_i^*|}{\phi_i(h)} \tag{2.19}$$

for all $n \in \mathbb{Z}^+$. The constant C may depend on the solution $x(n)$, *i.e.*, on the initial condition $x(0)$. Because of the compactness of the neighbourhood K of the equilibrium x^* this constant can be chosen so that (2.19) is valid for all solutions $x(n)$ of (2.12), (2.13) with $x(0) \in K$, and this completes the proof of the theorem.

The next theorem represents a generalization of Theorem 1. Namely, we replace the equality (2.4) by

$$|I_i(x_i(t_k))| \leq c|x_i(t_k) - x_i^*|, \quad i = \overline{1,m}, \quad k \in \mathbb{Z}^+, \tag{2.20}$$

for some positive constant c.

Now the equality (2.13) is replaced by

$$|x_i(n_k+1) - x_i(n_k)| \leq c|x_i(n_k) - x_i^*|, \quad i = \overline{1,m}, \quad k \in \mathbb{Z}^+. \tag{2.21}$$

Theorem 2 Let $h > 0$ satisfy (2.11). Suppose the assumptions **H1–H4** hold. Suppose further that

$$q\left(1 - \frac{\ln(1+c)}{\ln(1-\alpha'(h)\alpha)}\right) < 1. \tag{2.22}$$

Then there exists a number $\tilde{\rho} = \tilde{\rho}(h) < 1$ such that for any compact neighbourhood K of the equilibrium $\mathrm{x}^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$ there exists a constant $\beta \geq 1$ such that all solutions $x(n)$ of (2.12), (2.21) with $x(0) \in K$ satisfy the following inequality

$$\sum_{i=1}^{m} \frac{|x_i(n)-x_i^*|}{\phi_i(h)} \leq \beta\tilde{\rho}^n \sum_{i=1}^{m} \frac{|x_i(0)-x_i^*|}{\phi_i(h)}, \quad n \in \mathbb{Z}^+. \tag{2.23}$$

Proof. We need to just slightly modify the proof of Theorem 1 in case b). From the inequality (2.21) it follows that

$$|x_i(n_k+1)-x_i^*| \leq (1+c)|x_i(n_k)-x_i^*|, \quad i = \overline{1,m}, \quad k \in \mathbb{Z}^+,$$

and the inequality (2.18) should be replaced by

$$\sum_{i=1}^{m} \frac{|x_i(n)-x_i^*|}{\phi_i(h)} \leq \rho^{n-i(1,n)}(1+c)^{i(1,n)} \sum_{i=1}^{m} \frac{|x_i(0)-x_i^*|}{\phi_i(h)}, \quad n \in \mathbb{Z}^+. \tag{2.24}$$

For any $\eta > 0$ the inequality

$$i(1,n) \le (q+\eta)n$$

is satisfied for all n large enough. For such n we have

$$\rho^{n-i(1,n)}(1+c)^{i(1,n)} = \rho^{n-i(1,n)}\rho^{\frac{\ln(1+c)}{\ln\rho}i(1,n)} = \rho^{n-i(1,n)\left(1-\frac{\ln(1+c)}{\ln\rho}\right)}$$

$$\le \rho^{n-(q+\eta)n\left(1-\frac{\ln(1+c)}{\ln\rho}\right)} = \left\{\rho^{1-(q+\eta)\left(1-\frac{\ln(1+c)}{\ln\rho}\right)}\right\}^n.$$

By virtue of (2.22) we can choose $\eta > 0$ so that

$$(q+\eta)\left(1-\frac{\ln(1+c)}{\ln\rho}\right) < 1$$

and denote $\tilde{\rho} = \rho^{1-(q+\eta)\left(1-\frac{\ln(1+c)}{\ln\rho}\right)} < 1$. Inequality (2.24) becomes

$$\sum_{i=1}^{m}\frac{|x_i(n)-x_i^*|}{\phi_i(h)} \le \tilde{\rho}^n\sum_{i=1}^{m}\frac{|x_i(0)-x_i^*|}{\phi_i(h)}$$

for all n large enough. There exists a constant $\beta \ge 1$ such that the inequality (2.23) holds for all $n \in \mathbb{Z}^+$. This constant may depend on the solution $x(n)$, *i.e.*, on the initial condition $x(0)$. Because of the compactness of K we can choose β so that (2.23) is valid for all $x(0) \in K$.

In the formulation of the system (2.1) the neurons were implicitly assumed to process input and produce output instantaneously. It is known, however, that such instantaneous processing and delivery is not always true and there are significant time delays both in neural processing and axonal transmission. So system (2.1) was generalized in (Mohamad & Gopalsamy, 2000) by inserting time delays in neural networks. Accordingly, in (Akça et al., submitted) the following system

$$\frac{dx_i}{dt} = -a_ix_i(t) + \sum_{j=1}^{m} b_{ij}f_j(x_j(t-\tau_{ij})) + c_i, \quad t > 0, \quad t \ne t_k, \tag{2.25}$$

was considered, in which $i = \overline{1,m}$ and $\tau_{ij} \ge 0$ corresponds to the transmission delay for $i,j \in \{1,2,\dots,m\}$.

In (Akça et al., submitted) the solutions of (2.25) satisfying the impulsive conditions (2.2) with (2.4) and the initial conditions

$$x_i(s) = \psi_i(s), \ s \in [-\tau, 0], \ i = \overline{1,m}, \ \tau = \max_{i,j\in\{1,\dots,m\}}\{\tau_{ij}\},$$

with $\psi_i(s)$ continuous for $s \in [-\tau, 0]$ were proved to satisfy

$$\sum_{i=1}^{m}|x_i(t)-x_i^*| \le \beta e^{-\varepsilon t}\sum_{i=1}^{m}\left(\sup_{s\in[-\tau,0]}|x_i(s)-x_i^*|\right), \quad t > 0,$$

for some constants $\beta \ge 1$ and $\varepsilon > 0$.

The discrete-time analogue of (2.25) obtained in (Mohamad & Gopalsamy, 2000) is

$$x_i(n+1) = e^{-a_ih}x_i(n) + \phi_i(h)\sum_{j=1}^{m} b_{ij}f_j(x_j(n-\kappa_{ij})) + \phi_i(h)c_i, \quad i = \overline{1,m}, \tag{2.26}$$

for $n \in \mathbb{Z}_0^+$, where $\kappa_{ij} = [\tau_{ij}/h]$, with initial condition

$$x_i(\ell) = \psi_i(\ell), \ i = \overline{1,m}, \ \ell = \overline{-\kappa, 0}, \ \kappa = \max_{i,j\in\{1,\dots,m\}}\{\kappa_{ij}\}. \tag{2.27}$$

We take the equation (2.26) for $n \neq n_k$, i.e.,

$$(2.28)\qquad x_i(n+1) = e^{-a_i h} x_i(n) + \phi_i(h) \sum_{j=1}^{m} b_{ij} f_j(x_j(n-\kappa_{ij})) + \phi_i(h) c_i,$$
$$i = \overline{1,m}, \quad n \in \mathbb{Z}_0^+ \setminus \{n_1, n_2, \ldots\}.$$

Theorem} We again replace the impulsive condition (2.2) with (2.4) by (2.13). Our next result is the following Let $h > 0$ satisfy (2.11). Suppose the assumptions **H1**–**H4** hold and let κ_{ij} $i,j = \overline{1,m}$, denote nonnegative integers. Then there exist constants $\beta \geq 1$ and $\lambda > 1$ such that all solutions of (2.28), (2.13), (2.27) satisfy

$$(2.29)\qquad \sum_{i=1}^{m} \frac{|x_i(n) - x_i^*|}{\phi_i(h)} \leq \beta \lambda^{-n} \sum_{i=1}^{m} \sup_{\ell = -\kappa, 0} \frac{|\psi_i(\ell) - x_i^*|}{\phi_i(h)}, \quad n \in \mathbb{Z}^+,$$

provided that

$$(2.30)\qquad |1 - \gamma_{ik}| \leq e^{-a_i h}, \qquad i = \overline{1,m},$$

for all $k \in \mathbb{Z}^+$.

Proof. For $n \neq n_k$, $k \in \mathbb{Z}^+$, the proof follows that of (Mohamad & Gopalsamy, 2000, Theorem 4.2). From (2.28) it follows that

$$(2.31)\qquad |x_i(n+1) - x_i^*| \leq e^{-a_i h} |x_i(n) - x_i^*| + \phi_i(h) \sum_{j=1}^{m} |b_{ij}|\, L_j\, |x_j(n-\kappa_{ij}) - x_j^*|,$$
$$i = \overline{1,m}, \; n \in \mathbb{Z}_0^+ \setminus \{n_1, n_2, \ldots\}.$$

We consider functions $G_i \in C[0,\infty)$, $i = \overline{1,m}$, defined by

$$G_i(\lambda_i) = 1 - \lambda_i e^{-a_i h} - L_i \phi_i(h) \sum_{j=1}^{m} |b_{ji}| \lambda_i^{\kappa_{ji}+1}.$$

We have

$$G_i(1) = 1 - e^{-a_i h} - L_i \phi_i(h) \sum_{j=1}^{m} |b_{ji}| = \phi_i(h) \left(a_i - L_i \sum_{j=1}^{m} |b_{ji}| \right) > 0$$

for all $i = \overline{1,m}$ by virtue of condition **H4**.

By the continuity of $G_i(\cdot)$ there exist $\tilde{\lambda}_i > 1$, $i = \overline{1,m}$, such that $G_i(\lambda_i) > 0$ for $1 \leq \lambda_i \leq \tilde{\lambda}_i$, $i = \overline{1,m}$. Let $\tilde{\lambda} = \min\{\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_m\} > 1$. Then for $1 < \lambda \leq \tilde{\lambda}$ we have

$$(2.32)\qquad G_i(\lambda) = 1 - \lambda^{-a_i h} - L_i \phi_i(h) \sum_{j=1}^{m} |b_{ji}| \lambda^{\kappa_{ji}+1} \geq 0, \quad i = \overline{1,m}.$$

If we denote

$$(2.33)\qquad y_i(n) = \lambda^n \frac{|x_i(n) - x_i^*|}{\phi_i(h)}, \quad i = \overline{1,m}, \quad n \in \{-\kappa, -\kappa+1, \ldots, -1\} \cup \mathbb{Z}_0^+,$$

then from (2.31) we have

$$(2.34)\qquad y_i(n+1) \leq \lambda e^{-a_i h} y_i(n) + \sum_{j=1}^{m} |b_{ij}| L_j \phi_j(h) \lambda^{\kappa_{ij}+1} y_j(n-\kappa_{ij}),$$
$$n \in \mathbb{Z}_0^+ \setminus \{n_1, n_2, \ldots\},$$

for $i = \overline{1,m}$. Next consider a Lyapunov functional $V(n) = V(y_1, y_2, \ldots, y_m)(n)$ defined by

$$(2.35)\qquad V(n) = \sum_{i=1}^{m} \left\{ y_i(n) + \sum_{j=1}^{m} |b_{ij}| L_j \phi_j(h) \lambda^{\kappa_{ij}+1} \sum_{\ell = n-\kappa_{ij}}^{n-1} y_j(\ell) \right\}, \quad n \in \mathbb{Z}_0^+.$$

We note that $V(n) \geq 0$ for $n \in \mathbb{Z}_0^+$ and $V(0)$ is finite. We estimate the difference $V(n+1) - V(n)$ along the solutions of (2.28) for $n \neq n_k,\ k \in \mathbb{Z}^+$:

$$\begin{aligned} V(n+1) - V(n) &\leq \sum_{i=1}^{m} \left\{ (\lambda e^{-a_i h} - 1) y_i(n) + \sum_{j=1}^{m} |b_{ij}| L_j \phi_j(h) \lambda^{\kappa_{ij}+1} y_j(n) \right\} \\ &= -\sum_{i=1}^{m} \left\{ 1 - \lambda e^{-a_i h} - L_i \phi_i(h) \sum_{j=1}^{m} |b_{ji}| \lambda^{\kappa_{ji}+1} \right\} y_i(n). \end{aligned}$$

By virtue of (2.32) we have

$$V(n+1) \leq V(n) \qquad \text{for} \qquad n \in \mathbb{Z}_0^+ \setminus \{n_1, n_2, \ldots\}.$$

Next we find successively

$$\begin{aligned} |x_i(n_k+1) - x_i^*| &= |1 - \gamma_{ik}||x_i(n_k) - x_i^*|, \\ y_i(n_k+1) &= |1 - \gamma_{ik}|\lambda y_i(n_k), \end{aligned}$$

$$\begin{aligned} V(n_k+1) &= \sum_{i=1}^{m} \left\{ |1 - \gamma_{ik}|\lambda y_i(n_k) + \sum_{j=1}^{m} |b_{ij}| L_j \phi_j(h) \lambda^{\kappa_{ij}+1} \sum_{\ell = n_k+1-\kappa_{ij}}^{n_k} y_j(\ell) \right\} \\ &= \sum_{i=1}^{m} \left\{ \left(|1 - \gamma_{ik}|\lambda + L_i \phi_i(h) \sum_{j=1}^{m} |b_{ji}| \lambda^{\kappa_{ji}+1} \right) y_i(n_k) \right. \\ &\quad \left. + \sum_{j=1}^{m} |b_{ij}| L_j \phi_j(h) \lambda^{\kappa_{ij}+1} \left(\sum_{\ell = n_k - \kappa_{ij}}^{n_k - 1} y_j(\ell) - y_j(n_k - \kappa_{ij}) \right) \right\}. \end{aligned}$$

By definition

$$V(n_k) = \sum_{i=1}^{m} \left\{ y_i(n_k) + \sum_{j=1}^{m} |b_{ij}| L_j \phi_j(h) \lambda^{\kappa_{ij}+1} \sum_{\ell = n_k - \kappa_{ij}}^{n_k - 1} y_j(\ell) \right\},$$

thus

$$\begin{aligned} V(n_k+1) - V(n_k) &\leq \sum_{i=1}^{m} \left\{ |1 - \gamma_{ik}|\lambda - 1 + L_i \phi_i(h) \sum_{j=1}^{m} |b_{ji}| \lambda^{\kappa_{ji}+1} \right\} y_i(n_k) \\ &= -\sum_{i=1}^{m} \left\{ 1 - \lambda e^{-a_i h} - L_i \phi_i(h) \sum_{j=1}^{m} |b_{ji}| \lambda^{\kappa_{ji}+1} \right\} y_i(n_k) \\ &\quad + \lambda \sum_{i=1}^{m} \left(|1 - \gamma_{ik}| - e^{-a_i h} \right) y_i(n_k). \end{aligned}$$

By virtue of (2.32) and (2.30) we have $V(n_k+1) - V(n_k) \leq 0$ for $k \in \mathbb{Z}^+$ and, finally, $V(n+1) \leq V(n)$ for all $n \in \mathbb{Z}_0^+$, which implies that $V(n) \leq V(0)$ for $n \in \mathbb{Z}^+$.

Now the proof is completed as that of (Mohamad & Gopalsamy, 2000, Theorem 4.2). Namely, from (2.35) it follows that

$$\begin{aligned} \sum_{i=1}^{m} y_i(n) &\leq \sum_{i=1}^{m} \left\{ y_i(0) + \sum_{j=1}^{m} |b_{ij}| L_j \phi_j(h) \lambda^{\kappa_{ij}+1} \sum_{\ell=-\kappa_{ij}}^{-1} y_j(\ell) \right\} \\ &= \sum_{i=1}^{m} \left\{ y_i(0) + L_i \phi_i(h) \sum_{j=1}^{m} |b_{ji}| \lambda^{\kappa_{ji}+1} \sum_{\ell=-\kappa_{ji}}^{-1} y_i(\ell) \right\}, \quad n \in \mathbb{Z}^+, \end{aligned}$$

and hence, by virtue of (2.33), we have

$$\sum_{i=1}^{m} \frac{|x_i(n) - x_i^*|}{\phi_i(h)} \le \lambda^{-n} \sum_{i=1}^{m} \left\{ 1 + L_i\phi_i(h) \sum_{j=1}^{m} |b_{ji}|\kappa_{ji}\lambda^{\kappa_{ji}+1} \right\} \sup_{\ell=\overline{-\kappa,0}} \frac{|\psi_i(\ell) - x_i^*|}{\phi_i(h)},$$

which yields (2.29) with

$$\beta = \beta(h,\lambda) = \max_{i=\overline{1,m}} \left\{ 1 + L_i\phi_i(h) \sum_{j=1}^{m} |b_{ji}|\kappa_{ji}\lambda^{\kappa_{ji}+1} \right\} \ge 1.$$

In the model (2.25) the time delays were assumed discrete. While this assumption is not unreasonable, a more satisfactory hypothesis is that the time delays are continuously distributed over a certain duration of time. The system (2.25) was modified to a system of integro-differential equations (Akça et al., 1996; Akça et al., 2002; Mohamad & Gopalsamy, 2000) of the form

$$\frac{dx_i(t)}{dt} = -a_i x_i(t) + \sum_{j=1}^{m} b_{ij} f_j \left(\int_{-\infty}^{t} K_{ij}(t-s) x_j(s)\, ds \right) + c_i, \quad i = 1,2,\dots,m, \quad t > 0,$$

or

$$\begin{aligned} \frac{dx_i(t)}{dt} &= -a_i x_i(t) + \sum_{j=1}^{m} b_{ij} f_j \left(\int_{0}^{\infty} K_{ij}(s) x_j(t-s)\, ds \right) + c_i, \\ & \qquad i = 1,2,\dots,m, \qquad t > 0, \end{aligned} \tag{2.36}$$

where for $i,j = \overline{1,m}$ the delay kernels $K_{ij}(s)$ were assumed to satisfy the following conditions:

- $K_{ij} : [0,\infty) \to [0,\infty)$ are bounded and continuous.
- $\int_0^\infty K_{ij}(s)\, ds = 1$.
- There exists a positive number μ such that $\int_0^\infty K_{ij}(s) e^{\mu s}\, ds < \infty$.

For an integro-differential equation an impulsive condition including both the functional value and its integral also seems natural. Thus in (Akça et al., submitted) we took the impulse conditions in the form

$$\Delta x_i(t_k) = I_i(x_i(t_k)) = B_{ik} x_i(t_k) + \int_{t_{k-1}}^{t_k} c_{ik}(s) x_i(s)\, ds + \alpha_{ik}, \quad k \in \mathbb{Z}^+, \tag{2.37}$$

where $t_k > t_0 = 0$ and $c_{ik} : [t_{k-1}, t_k] \to \mathbb{R}$ are measurable functions, essentially bounded on the respective interval, B_{ik} and α_{ik} are some real constants. For more details about impulse conditions of this and more general form see (Akça et al., 1996) and reference therein.

The initial conditions associated with the system (2.36), (2.37) are given by

$$x_i(s) = \psi_i(s), \quad s \in (-\infty, 0], \qquad i = \overline{1,m},$$

where $\psi_i(s)$ are bounded and continuous on $(-\infty,0]$. In (Akça et al., submitted) the solutions of system (2.36), (2.37) were proved to satisfy the estimate

$$\sum_{i=1}^{m} |x_i(t) - x_i^*| \le \beta e^{-\epsilon t} \prod_{k=1}^{i(0,t)} \{1 + B_k(1 + \varphi(\epsilon, t_k - t_{k-1}))\} \sum_{i=1}^{m} \sup_{w \in (-\infty,0]} |x_i(w) - x_i^*| \tag{2.38}$$

for all $t > 0$ and some $\epsilon > 0$, $\beta \ge 1$, where $\mathbf{x}^* = (x_1^*, \dots, x_m^*)$ is an equilibrium whose components satisfy (2.3) as well as

$$\left(B_{ik} + \int_{t_{k-1}}^{t_k} c_{ik}(s)\, ds \right) x_i^* + \alpha_{ik} = 0, \quad i = \overline{1,m}, \quad k \in \mathbb{Z}^+, \tag{2.39}$$

$$B_k = \max_{i=\overline{1,m}} \max \left\{ |B_{ik}|, \|c_{ik}\|_{L_\infty(t_{k-1},t_k)} \right\},$$

and the function $\varphi(\varepsilon, z)$ is defined for $\varepsilon, z > 0$ by $\varphi(\varepsilon, z) = (e^{\varepsilon z} - 1)/\varepsilon$.

In the discrete-time analogue of (2.36) the integral term $\int_0^\infty K_{ij}(s)x_j(t-s)\,ds$, $i,j = \overline{1,m}$, is replaced by a sum of the form $\sum_{p=1}^{\infty} K_{ij}(p)x_j(n-p)$, where $n = [t/h]$, $p = [s/h]$, by an abuse of notation $K_{ij}(p)$ stands for $K_{ij}(ph)$ and $x_j(n-p)$ for $x_j((n-p)h)$, and the discrete kernels $K_{ij}(\cdot)$, $i,j = \overline{1,m}$, satisfy the following conditions:

H5.: $K_{ij}(p) \in [0,\infty)$ and is bounded for $p \in \mathbb{Z}^+$.

H6.: $\sum_{p=1}^{\infty} K_{ij}(p) = 1$.

H7.: There exists a number $\nu > 1$ such that $\sum_{p=1}^{\infty} K_{ij}(p)\nu^p < \infty$.

The discrete-time analogue of (2.36) obtained in (Mohamad & Gopalsamy, 2000) in the same manner as (2.9) was derived from (2.1) is

$$(2.40) \qquad \begin{aligned} x_i(n+1) &= e^{-a_i h}x_i(n) + \phi_i(h)\sum_{j=1}^{n} b_{ij} f_i\left(\sum_{p=1}^{\infty} K_{ij}(p)x_j(n-p)\right) \\ &\quad + \phi_i(h)c_i, \quad i = \overline{1,m}, \end{aligned}$$

and it is supplemented with initial values of the form

$$(2.41) \qquad x_i(r) = \psi_i(r), \qquad r \in \mathbb{Z}_0^-,$$

and the sequence $\{\psi_i(r)\}_{r=-\infty}^{0}$ is bounded for all $i = \overline{1,m}$. We take the equation (2.40) for $n \neq n_k$, *i.e.*,

$$(2.42) \qquad \begin{aligned} x_i(n+1) &= e^{-a_i h}x_i(n) + \phi_i(h)\sum_{j=1}^{n} b_{ij} f_i\left(\sum_{p=1}^{\infty} K_{ij}(p)x_j(n-p)\right) + \phi_i(h)c_i, \\ & i = \overline{1,m}, \quad n \in \mathbb{Z}_0^+ \setminus \{n_1, n_2, \ldots\}. \end{aligned}$$

The impulsive conditions (2.37) are approximated by

$$(2.43) \qquad x_i(n_k+1) - x_i(n_k) = \sum_{\ell=n_{k-1}+1}^{n_k} B_{ik\ell}x_i(\ell) + \alpha_{ik}, \quad i = \overline{1,m}, \quad k \in \mathbb{Z}^+,$$

where, for convenience, $n_0 = -1$ and the constants $B_{ik\ell}$ satisfy relations given below.

An equilibrium $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$ of (2.42), (2.43) must satisfy (2.3) and

$$(2.44) \qquad \sum_{\ell=n_{k-1}+1}^{n_k} B_{ik\ell}x_i^* + \alpha_{ik} = 0, \quad i = \overline{1,m}, \quad k \in \mathbb{Z}^+.$$

By a comparison of (2.44) with (2.39) we find

$$\sum_{\ell=n_{k-1}+1}^{n_k} B_{ik\ell} = B_{ik} + \int_{t_{k-1}}^{t_k} c_{ik}(s)\,ds, \quad i = \overline{1,m}, \ k \in \mathbb{Z}^+.$$

We denote

$$B_k = \max_{i=\overline{1,m}} \max_{n_{k-1}+1 \le \ell \le n_k} |B_{ik\ell}|, \qquad k \in \mathbb{Z}^+.$$

Our result is Let $h > 0$ satisfy (2.11) and let conditions **H1**–**H7** hold. Suppose there exists a point $\mathbf{x}^* = (x_1^*, x_2^*, \dots, x_m^*)^T$ satisfying the equalities (2.3) and (2.44). Then there exist constants $\beta \geq 1$ and $\lambda > 1$ such that all solutions of (2.42), (2.43), (2.41) satisfy

$$\sum_{i=1}^{m} \frac{|x_i(n)-x_i^*|}{\phi_i(h)} \leq \beta\lambda^{-n} \prod_{k=1}^{i(1,n)} (1+\lambda+B_k\chi(\lambda, n_k - n_{k-1})) \sum_{i=1}^{m} \sup_{r\in\mathbb{Z}_0^-} \frac{|\psi_i(r)-x_i^*|}{\phi_i(h)} \tag{2.45}$$

for $n \in \mathbb{Z}^+$, where $\chi(\lambda, n) = \sum_{\ell=1}^{n} \lambda^\ell$. *Proof.* It is clear that there can exist at most one equilibrium point. By our assumptions it does exist.

For $n \neq n_k$, $k \in \mathbb{Z}^+$, we follow the proof of (Mohamad & Gopalsamy, Theorem 4.3). We have from (2.42) that

$$|x_i(n+1)-x_i^*| \leq e^{-a_i h}|x_i(n)-x_i^*| + \phi_i(h)\sum_{j=1}^{m} |b_{ij}|L_j \sum_{p=1}^{\infty} K_{ij}(p)|x_j(n-p)-x_j^*|, \tag{2.46}$$

where $i = \overline{1,m}$, $n \neq n_k$.

Let us consider the functions $G_i : [0,\nu] \to \mathbb{R}$, $i = \overline{1,m}$, defined by

$$G_i(\lambda_i) = 1 - \lambda_i e^{-a_i h} - L_i\phi_i(h)\sum_{j=1}^{m} |b_{ji}| \sum_{p=1}^{\infty} K_{ji}(p)\lambda_i^{p+1}.$$

By virtue of condition **H7** the functions $G_i(\cdot)$, $i = \overline{1,m}$, are well defined and continuous on $[0,\nu]$. Moreover,

$$G_i(1) = 1 - e^{-a_i h} - L_i\phi_i(h)\sum_{j=1}^{m} |b_{ji}| \sum_{p=1}^{\infty} K_{ji}(p) = \phi_i(h)\left(a_i - L_i\sum_{j=1}^{m}|b_{ji}|\right) > 0$$

by condition **H4**. By the continuity of $G_i(\cdot)$ there exist numbers $\tilde{\lambda}_i \in (1,\nu]$, $i = \overline{1,m}$, such that $G_i(\lambda_i) > 0$ for $\lambda_i \in (1, \tilde{\lambda}_i]$. Let $\tilde{\lambda} = \min\{\tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_m\}$. Then for $1 < \lambda \leq \tilde{\lambda}$ we have

$$G_i(\lambda) = 1 - \lambda e^{-a_i h} - L_i\phi_i(h)\sum_{j=1}^{m} |b_{ji}| \sum_{p=1}^{\infty} K_{ji}(p)\lambda^{p+1} > 0, \quad i = \overline{1,m}. \tag{2.47}$$

Now denote

$$y_i(t) = \lambda^n \frac{|x_i(n) - x_i^*|}{\phi_i(h)}, \quad i = \overline{1,m},\ n \in \mathbb{Z}.$$

Then from (2.46) for $n \neq n_k$ we derive

$$y_i(n+1) \leq \lambda e^{-a_i h} y_i(n) + \sum_{j=1}^{m} |b_{ij}|L_j\phi_i(h) \sum_{p=1}^{\infty} K_{ij}(p)\lambda^{p+1} y_j(n-p), \quad i = \overline{1,m}.$$

We define a Lyapunov functional $V(\cdot)$ by

$$V(n) = \sum_{i=1}^{m}\left\{ y_i(n) + \sum_{j=1}^{m} |b_{ij}|L_j\phi_j(h) \sum_{p=1}^{\infty} K_{ij}(p)\lambda^{p+1} \sum_{r=n-p}^{n-1} y_j(r)\right\}, \quad n \in \mathbb{Z}_0^+.$$

It is easy to see that $V(n) \geq 0$ for $n \in \mathbb{Z}_0^+$ and $V(0) < \infty$ by **H7**.

We can now estimate the difference $V(n+1)-V(n)$ along the solutions of (2.42) for $n \neq n_k$ as follows:

$$V(n+1)-V(n) \le \sum_{i=1}^{m}\left\{(\lambda e^{-a_i h}-1)y_i(n)+\sum_{j=1}^{m}|b_{ij}|L_j\phi_j(h)\sum_{p=1}^{\infty}K_{ij}(p)\lambda^{p+1}y_j(n)\right\}$$

$$= -\sum_{i=1}^{m}\left\{1-\lambda e^{-a_i h}-L_i\phi_i(h)\sum_{j=1}^{m}|b_{ji}|\sum_{p=1}^{\infty}K_{ji}(p)\lambda^{p+1}\right\}y_i(n)\le 0$$

by virtue of (2.47). It follows that

$$V(n)\le V(n_{k-1}+1) \quad \text{for} \quad n_{k-1}<n\le n_k,\ k\in\mathbb{Z}^+. \tag{2.48}$$

Further on, making use of the equalities (2.43) and (2.44), for any n_k, $k\in\mathbb{Z}^+$, we successively find

$$|x_i(n_k+1)-x_i^*| \le |x_i(n_k)-x_i^*|+\sum_{\ell=n_{k-1}+1}^{n_k}|B_{ik\ell}|\,|x_i(\ell)-x_i^*|$$

$$\le (1+B_k)|x_i(n_k)-x_i^*|+B_k\sum_{\ell=n_{k-1}+1}^{n_k-1}|x_i(\ell)-x_i^*|,$$

$$y_i(n_k+1) \le (1+B_k)\lambda y_i(n_k)+B_k\sum_{\ell=n_{k-1}+1}^{n_k-1}\lambda^{n_k+1-\ell}y_i(\ell), \quad i=\overline{1,m},$$

and thus

$$V(n_k+1) \le \sum_{i=1}^{m}\left\{(1+B_k)\lambda y_i(n_k)+B_k\sum_{\ell=n_{k-1}+1}^{n_k-1}\lambda^{n_k+1-\ell}y_i(\ell)\right.$$

$$\left.+\sum_{j=1}^{m}|b_{ij}|L_j\phi_j(h)\sum_{p=1}^{\infty}K_{ij}(p)\lambda^{p+1}\sum_{r=n_k+1-p}^{n_k}y_j(r)\right\}$$

$$=\sum_{i=1}^{m}\left\{\left[(1+B_k)\lambda+L_i\phi_i(h)\sum_{j=1}^{m}|b_{ji}|\sum_{p=1}^{\infty}K_{ji}(p)\lambda^{p+1}\right]y_i(n_k)\right.$$

$$\left.+B_k\sum_{\ell=n_{k-1}+1}^{n_k-1}\lambda^{n_k+1-\ell}y_i(\ell)+L_i\phi_i(h)\sum_{j=1}^{m}|b_{ji}|\sum_{p=1}^{\infty}K_{ji}(p)\lambda^{p+1}\sum_{r=n_k+1-p}^{n_k-1}y_i(r)\right\}.$$

From the definition of $V(\cdot)$ it follows that

$$(1+\lambda B_k)V(n_k)=\sum_{i=1}^{m}\Big\{(1+\lambda B_k)y_i(n_k)$$

$$+\ (1+\lambda B_k)L_i\phi_i(h)\sum_{j=1}^{m}|b_{ji}|\sum_{p=1}^{\infty}K_{ji}(p)\lambda^{p+1}\sum_{r=n_k-p}^{n_k-1}y_i(r)\Big\},$$

thus

$$V(n_k+1)\le(1+\lambda B_k)V(n_k)$$

$$-\sum_{i=1}^{m}\left\{1-\lambda e^{-a_i h}-L_i\phi_i(h)\sum_{j=1}^{m}|b_{ji}|\sum_{p=1}^{\infty}K_{ji}(p)\lambda^{p+1}\right\}y_i(n_k)$$

$$+\ \lambda\sum_{i=1}^{m}y_i(n_k)+B_k\sum_{i=1}^{m}\sum_{\ell=n_{k-1}+1}^{n_k-1}\lambda^{n_k+1-\ell}y_i(\ell)$$

for $k \in \mathbb{Z}^+$. By virtue of (2.47) and (2.48) we find

$$V(n_k+1) \le \left(1+\lambda+B_k \sum_{\ell=1}^{n_k-n_{k-1}} \lambda^\ell\right) V(n_{k-1}+1)$$

for $k \in \mathbb{Z}^+$ $(n_0 = -1)$, *i.e.*,

$$V(n_k+1) \le \{1+\lambda+B_k\chi(\lambda, n_k-n_{k-1})\}V(n_{k-1}+1).$$

If we combine the last inequality and (2.48), we obtain

$$V(n) \le \prod_{k=1}^{i(1,n)} \{1+\lambda+B_k\chi(\lambda, n_k-n_{k-1})\}\, V(0), \quad n \in \mathbb{Z}^+,$$

which implies

$$\begin{aligned}
\sum_{i=1}^m \frac{|x_i(n)-x_i^*|}{\phi_i(h)} &\le \lambda^{-n} \prod_{k=1}^{i(1,n)} \{1+\lambda+B_k\chi(\lambda, n_k-n_{k-1})\} \sum_{i=1}^m \left\{ \frac{|x_i(0)-x_i^*|}{\phi_i(h)} \right. \\
&\quad + \left. \sum_{j=1}^m |b_{ij}| L_j \phi_j(h) \sum_{p=1}^\infty K_{ij}(p) \lambda^{p+1} \sum_{r=-p}^{-1} \frac{|x_i(r)-x_i^*|}{\phi_i(h)} \right\} \\
&\le \beta\lambda^{-n} \prod_{k=1}^{i(1,n)} \{1+\lambda+B_k\chi(\lambda, n_k-n_{k-1})\} \sum_{i=1}^m \sup_{r\in\mathbb{Z}_0^-} \frac{|\psi_i(r)-x_i^*|}{\phi_i(h)}
\end{aligned}$$

for $n \in \mathbb{Z}^+$, where

$$\beta = \beta(h, \lambda) = \max_{i=\overline{1,m}} \left\{ 1 + L_i\phi_i(h) \sum_{j=1}^m |b_{ji}| \sum_{p=1}^\infty K_{ji}(p) p \lambda^{p+1} \right\}.$$

This completes the proof of Theorem 4.

Just as the estimate (2.38) in the continuous-time case, the estimate (2.45) does not imply in general any sort of stability of the equilibrium point $\mathbf{x}^*$ of system (2.42), (2.43). In order to prove stability we have to impose some additional conditions on the moments and amplitudes of the impulse effects.

H8.: $\sup_{k\in\mathbb{Z}^+} (n_k - n_{k-1}) < \infty$.

Suppose that

$$N > n_k - n_{k-1} \quad \forall k \in \mathbb{Z}^+. \tag{2.49}$$

Since $\lim_{\lambda\to 1+} \chi(\lambda, n) = n$, we can choose $\lambda \in (1, \tilde{\lambda}]$ so that

$$\chi(\lambda, n_k - n_{k-1}) \le N \quad \forall k \in \mathbb{Z}^+. \tag{2.50}$$

H9.: $B = \sup_{k\in\mathbb{Z}^+} B_k(1+N) < \infty$.

Corollary 2.1. *Let all conditions of Theorem 4 hold. Suppose that the impulse effects* (2.43) *satisfy the assumptions* **H8, H9**, *where* N *is defined by* (2.49). *If* $\lambda \in (1, \tilde{\lambda}]$ *is chosen to satisfy* (2.50) *and*

$$\lambda > (1+\lambda+B)^q, \tag{2.51}$$

then for any number λ^* *satisfying*

$$1 < \lambda^* < \lambda^{1-\frac{\ln(1+\lambda+B)}{\ln\lambda}q} \tag{2.52}$$

and any compact neighbourhood K of the equilibrium $\mathbf{x}^* = (x_1^*, x_2^*, \dots, x_m^*)^T$ there exists a constant $\beta \geq 1$ such that all solutions $x(n)$ of (2.42), (2.43) *with $x(r) \in K$, $r \in \mathbb{Z}_0^-$ satisfy the inequality*

$$\sum_{i=1}^{m} \frac{|x_i(n) - x_i^*|}{\phi_i(h)} \leq \beta \lambda^{*-n} \sum_{i=1}^{m} \sup_{r \in \mathbb{Z}_0^-} \frac{|x_i(r) - x_i^*|}{\phi_i(h)} \quad \forall n \in \mathbb{Z}^+. \tag{2.53}$$

Proof. First let us note that the conditions of Corollary 1 can be satisfied. Indeed, let us suppose that the moments of impulse effect satisfy the condition **H8**. Let us fix λ satisfying (2.50). For given constants B_k, $k \in \mathbb{Z}^+$, and B defined by the condition **H9**, we can take $h > 0$ so small that $q = ph$ satisfies the inequality (2.51).

Now suppose that all conditions of Corollary 1 hold. By virtue of **H8**, (2.49), (2.50) and **H9** inequality (2.45) takes on the form

$$\sum_{i=1}^{m} \frac{|x_i(n) - x_i^*|}{\phi_i(h)} \leq \beta \lambda^{-n} (1 + \lambda + B)^{i(1,n)} \sum_{i=1}^{m} \sup_{r \in \mathbb{Z}_0^-} \frac{|x_i(r) - x_i^*|}{\phi_i(h)}.$$

For any $\eta > 0$ the inequality $i(1,n) \leq (q + \eta)n$ is satisfied for all n large enough. Then we have

$$\lambda^{-n}(1 + \lambda + B)^{i(1,n)} \leq \lambda^{-n} \lambda^{\frac{\ln(1+\lambda+B)}{\ln \lambda}(q+\eta)n} = \{\lambda^{1 - \frac{\ln(1+\lambda+B)}{\ln \lambda}(q+\eta)}\}^{-n}.$$

According to (2.51) $\frac{\ln(1+\lambda+B)}{\ln \lambda} q < 1$. We take $\eta > 0$ so that $\frac{\ln(1+\lambda+B)}{\ln \lambda}(q + \eta) < 1$ and denote

$$\lambda^* = \lambda^{1 - \frac{\ln(1+\lambda+B)}{\ln \lambda}(q+\eta)}. \tag{2.54}$$

Thus (2.52) is valid (or, conversely, for any λ^* satisfying (2.52) we can choose $\eta > 0$ so that (2.54) holds) and the estimate (2.53) is valid for all n large enough. If we increase the constant β, we can provide the validity of the estimate (2.53) for all $n \in \mathbb{Z}^+$. The constant β may still depend on the solution $x(n)$ of system (2.42), (2.43), *i.e.*, on the initial function $\psi(r)$. Because of the compactness of K we can choose β so that (2.53) is valid for all solutions $x(n)$ such that $x(r) \in K$ for $r \in \mathbb{Z}_0^-$.

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REFERENCES

1. Akça, H., Berezansky, L., & Braverman, E. (1996). On linear integro-differential equations with integral impulsive conditions. *ZAA J. for Analysis and its Applications*, v.15, pp. 709-727.
2. Akça, H., Boucherif, A., & Covachev, V. (2002). Impulsive functional-differential equations with nonlocal conditions. *Int. J. Math. and Mathematical Sciences*, v.29, pp. 251-256.
3. Akça, H., Alassar, R., Covachev, V., Covacheva, Z., & Al-Zahrani, E. (submitted). Continuous-time additive Hopfield-type neural networks with impulses.
4. Covachev, V., Akça, H., & Yeniçerioğlu, F. (2001). Difference approximations for impulsive differential equations, *Applied Mathematics and Computation*, v.121, pp. 383-390.
5. Meinardus, G., & Nurnberger, G. (Eds.) (1985). *Delay Equations, Approximation and Application*, Boston: Birkhäuser.

6. Mickens, R. E. (1994). *Nonstandard Finite Difference Models of Differential Equations*, Singapore: World Scientific.
7. Mohamad, S., & Gopalsamy, K. (2000). Dynamics of a class of discrete-time neural networks and their continuous-time counterparts, *Mathematics and Computers in Simulation*, v.53, pp. 1–39.
8. Mohamad, S. (2000). Continuous and Discrete Dynamical Systems with Applications, *Ph. D. Thesis, The Flinders University of South Australia.*
9. Prufer, M. (1985). Turbulence in multistep methods for initial value problems, *SIAM J. Appl. Math.*, v.45, pp. 32-69.